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ORIGINAL ARTICLE

The metrizability of L-topological groups

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Abstract In this study, we study the metrizability of the notion of L-topological group defined by Ahsanullah 1988. We show that for any (separated) L-topological group there is an L-pseudo-metric (L-metric), in sense of Gähler which is defined using his notion of L-real numbers, compatible with the L-topology of this (separated) L-topological group. That is, any (separated) L-topological group is pseudo-metrizable (metrizable).

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1. Introduction

The notion of L-real numbers is defined and studied by S. Gähler and W. Gähler in [1]. $\mathbf{R}_L$ denotes the set of all L-real numbers. The subset $\mathbf{R}_L^*$ of $\mathbf{R}_L$ of all positive L-real numbers is used to define the L-pseudo-metric (L-metric) on a set X, by the same authors in [1], as a mapping of the cartesian product $X \times X$ to $\mathbf{R}_L^*$ which satisfying similar conditions to the conditions of the usual metric. In this paper we study the metrizability, using the L-pseudo-metric (L-metric) in sense of [1], of a notion of L-topological group which is introduced in [2] and studied in [3]. This L-topological groups is defined as a group equipped with an L-topology such that both the binary operation and the unary operation of the inverse are L-continuous with respect to this L-topology.

In this paper, using the uniformizability of L-topological groups introduced by the authors in [4], we show that any (separated) L-topological group is pseudo-metrizable (metrizable). In [4] is used the L-uniform structures which are defined in [5] on a set X, in a similar way to the usual case, as L-filters on $X \times X$.

In Section 2 of this paper we recall some results on L-filters, L-real numbers defined by Gähler in [1,6–8], and some separation axioms defined by the authors in [9–12].

Sections 3 and 4 introduce and show some results on L-metric and L-uniform spaces, respectively, which are needed to show the metrizability of L-topological groups. We will use the notion of L-topogenous structure [13].

In Section 5 we show that the L-pseudo-metric (L-metric), in sense of [1], induces the L-topology of a (separated) L-topological group, that is, any (separated) L-topological group is pseudo-metrizable (metrizable).

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2. On L-filters

Recall here some ideas concerning L-filters needed in the paper. Denote by L^X the set of all L-subsets of a non-empty set X, where L is a complete chain with different least and greatest elements 0 and 1, respectively [14]. Let $L_0 = L\backslash\{0\}$ and $L_1 = L\backslash\{1\}$. For each L-set $\lambda \in L^X$, let λ' denote the complement of λ, defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the L-subset x_α of X whose value α at x and 0 otherwise is called an L-point in X and the constant L-subset of X with value α will be denoted by $\bar{\alpha}$.

L-filters. By an L-filter on a non-empty set X we mean [7] a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\bar{\alpha}) \leqslant \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$, and also $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all λ, $\mu \in L^X$. $\mathcal{M}$ is called *homogeneous* [7] if $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are L-filters on X, $\mathcal{M}$ is called *finer than* $\mathcal{N}$, denoted by $\mathcal{M} \succ \mathcal{N}$, provided $\mathcal{M}(\lambda) \geqslant \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$. By $\mathcal{M} \not\succ \mathcal{N}$ we mean that $\mathcal{M}$ is not finer than $\mathcal{N}$. Since L is a complete chain, then

$\mathcal{M} \not\succ \mathcal{N} \iff$ there is $f \in L^X$ such that $\mathcal{M}(f) < \mathcal{N}(f)$.

Let $\mathcal{F}_L X$ denote the set of all L-filters on X, $f: X \to Y$ a mapping and $\mathcal{M}, \mathcal{N}$ are L-filters on X, Y, respectively. Then the *image* of $\mathcal{M}$ and the *preimage* of $\mathcal{N}$ with respect to f are the L-filters $\mathcal{F}_L f(\mathcal{M})$ on Y and $\mathcal{F}_L^{-} f(\mathcal{N})$ on X defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^{-} f(\mathcal{N})(\lambda) = \bigvee_{\mu\circ f \leqslant \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f: X \to Y$ and each L-filter $\mathcal{N}$ on Y, for which the preimage $\mathcal{F}_L^{-} f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^{-} f(\mathcal{N})) \succ \mathcal{N}$. Moreover, for each L-filter $\mathcal{M}$ on X, the inequality $\mathcal{M} \succ \mathcal{F}_L^{-} f(\mathcal{F}_L f(\mathcal{M}))$ holds [7].

For any set A of L-filters on X, the infimum $\bigwedge_{\mathcal{M}\in A}\mathcal{M}$, with respect to the finer relation on L-filters, does not exist in general. The infimum $\bigwedge_{\mathcal{M}\in A}\mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n) \leqslant \sup(\lambda_1 \wedge \cdots \wedge \lambda_n)$ for all $\lambda_1, \ldots, \lambda_n \in L^X$ [6]. If the infimum of A exists, then for each $\lambda \in L^X$ and n as a positive integer we have

$$\left(\bigwedge_{\mathcal{M}\in A}\mathcal{M}\right)(\lambda) = \bigvee_{\substack{\lambda_1\wedge\cdots\wedge\lambda_n \leqslant \lambda,\\ \mathcal{M}_1,\ldots,\mathcal{M}_n \in A}} (\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n)).$$

By a *filter* on X we mean a non-empty subset $\mathcal{F}$ of L^X which does not contain $\bar{0}$ and closed under finite infima and super sets [15]. For each L-filter $\mathcal{M}$ on X, the subset $\alpha - \mathrm{pr}\ \mathcal{M}$ of L^X defined by: $\alpha - \mathrm{pr}\ \mathcal{M} = \{\lambda \in L^X | \mathcal{M}(\lambda) \geqslant \alpha\}$ is a filter on X.

A family $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ of non-empty subsets of L^X is called *valued L-filter base* on X [7] if the following conditions are fulfilled:

(V1) $\lambda \in \mathcal{B}_\alpha$ implies $\alpha \leqslant \sup\lambda$.

(V2) For all $\alpha, \beta \in L_0$ and all L-sets $\lambda \in \mathcal{B}_\alpha$ and $\mu \in \mathcal{B}_\beta$, if even $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geqslant \alpha \wedge \beta$ and an L-set $\nu \leqslant \lambda \wedge \mu$ such that $\nu \in \mathcal{B}_\gamma$.

Each valued L-filter base $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ on a set X defines an L-filter $\mathcal{M}$ on X by: $\mathcal{M}(\lambda) = \bigvee_{\mu\in\mathcal{B}_\alpha, \mu\leqslant\lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each L-filter $\mathcal{M}$ can be generated by many valued L-filter bases, and among them the greatest one $(\alpha - \mathrm{pr}\ \mathcal{M})_{\alpha\in L_0}$.

L-neighborhood filters. In the following, the topology in sense of [16,17] will be used which will be called L-topology. int_τ and cl_τ denote the interior and the closure operators with respect to the L-topology τ, respectively. For each L-topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by: $\mathcal{N}(x)(\lambda) = \mathrm{int}_\tau\lambda(x)$ for all $\lambda \in L^X$ is an L-filter on X, called the *L-neighborhood filter* of the space (X, τ) at x, and the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous L-filter on X. The L-neighborhood filters fulfill the following conditions:

(N1) $\dot{x} \succ \mathcal{N}(x)$ holds for all $x \in X$;

(N2) $(\mathcal{N}(x))(\mathrm{int}_\tau f) = (\mathcal{N}(x))(f)$ for all $x \in X$ and $f \in L^X$.

Let (X, τ) and (Y, σ) be two L-topological spaces. Then the mapping $f: (X, \tau) \to (Y, \sigma)$ is called *L-continuous* (or *(τ, σ)-continuous*) provided $\mathrm{int}_\sigma\mu \circ f \leqslant \mathrm{int}_\tau(\mu \circ f)$ for all $\mu \in L^Y$ [8].

The L-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L-filter on X defined, by the authors in [10], by means of $\mathcal{N}(x), x \in F$ as: $\mathcal{N}(F) = \bigvee_{x\in F}\mathcal{N}(x)$. The L-filter $\dot{F}$ is defined by: $\dot{F} = \bigvee_{x\in F}\dot{x}$. $\dot{F} \succ \mathcal{N}(F)$ holds for all subsets F of X. Recall also here the L-filter $\dot{\lambda}$ and the L-neighborhood filter $\mathcal{N}(\lambda)$ at an L-subset λ of X which are defined by

$$\dot{\lambda} = \bigvee_{0<\lambda(x)} \dot{x} \quad \text{and} \quad \mathcal{N}(\lambda) = \bigvee_{0<\lambda(x)} \mathcal{N}(x), \tag{2.1}$$

respectively. $\dot{\lambda} \succ \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$ [18].

L-real numbers. By an L-real number is meant [1] a convex, normal, compactly supported and upper semi-continuous L-subset of the set of all real numbers $\mathbf{R}$. The set of all L-real numbers is denoted by $\mathbf{R}_L$. $\mathbf{R}$ is canonically embedded into $\mathbf{R}_L$, identifying each real number a with the crisp L-number $a^\sim$ defined by $a^\sim(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive L-real numbers is defined and denoted by: $\mathbf{R}_L^* = \{x \in \mathbf{R}_L | x(0) = 1 \text{ and } 0^\sim \leqslant x\}$ and let $I_L = \{x \in \mathbf{R}_L^* | x \leqslant 1^\sim\}$, where $I = [0, 1]$ is the real unit interval. Note that we mean here by $\leqslant$ the binary operation on $\mathbf{R}_L$ defined by

$$x \leqslant y \iff x_{\alpha_1} \leqslant y_{\alpha_1} \quad \text{and} \quad x_{\alpha_2} \leqslant y_{\alpha_2}$$

for all $x, y \in \mathbf{R}_L$ where $x_{\alpha_1} = \inf\{z \in \mathbf{R} | x(z) \geqslant \alpha\}$ and $x_{\alpha_2} = \sup\{z \in \mathbf{R} | x(z) \geqslant \alpha\}$ for all $x \in \mathbf{R}_L$ and for all $\alpha \in L_0$. It is shown in [7] that the class $\{R_\delta|_{I_L} | \delta \in I\} \cup \{R^\delta|_{I_L} | \delta \in I\} \cup \{0^\sim|_{I_L}\}$ is a base for an L-topology $\mathfrak{I}$ on I_L, where R^δ and R_δ are the L-subsets of $\mathbf{R}_L$ defined by $R_\delta(x) = \bigvee_{\alpha>\delta} x(\alpha)$ and $R^\delta(x) = (\bigvee_{\alpha\geqslant\delta} x(\alpha))'$ for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$ and note that $R_\delta|_{I_L}, R^\delta|_{I_L}$ are the restrictions of R_δ, R^δ on I_L, respectively. Recall also that $x \pm y$ are L-real numbers defined by $(x \pm y)(\xi) = \bigvee_{\eta,\zeta\in\mathbf{R},\, \eta\pm\zeta=\xi} (x(\eta) \wedge y(\zeta))$ for all $\xi \in \mathbf{R}$. $(\mathbf{R}_L, +)$ is a commutative semi group with identity element $0^\sim$. The positive part x^+ of an L-real number x is defined as $x^+ = 0^\sim \vee x$, where

$$x - x = 0^\sim, \quad (x + y)^+ \leqslant x^+ + y^+. \tag{2.2}$$

GT_i-spaces. An L-topological space (X, τ) is called [9,11]:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\succ \mathcal{N}(y)$ or $\dot{y} \not\succ \mathcal{N}(x)$.
(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\succ \mathcal{N}(y)$ and $\dot{y} \not\succ \mathcal{N}(x)$.

(3) *completely regular* if for all $x \notin F$ and $F = \mathrm{cl}_\tau F$, there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \mathfrak{I})$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all $y \in F$.
(4) $GT_{3\frac{1}{2}}$ (or *L-Tychonoff*) if it is GT_1 and completely regular.

Proposition 2.1 (9–12). *Every GT_i-space is GT_{i-1}-space for all i = 1, 2, 3, 4, 5, 6. Moreover, the implications between GT_3-spaces, $GT_{3\frac{1}{2}}$-spaces and GT_4-spaces goes as expected.*

3. Some results on L-metric spaces

A mapping $\varrho : X \times X \to \mathbf{R}_L^*$ is called an L-metric [1] on X if the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^\sim$ if and only if $x = y$
(2) $\varrho(x, y) = \varrho(y, x)$
(3) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$.

If $\varrho : X \times X \to \mathbf{R}_L^*$ satisfies the conditions (2) and (3) and the following condition:

(1)′ $\varrho(x, y) = 0^\sim$ if $x = y$

then it is called an *L-pseudo-metric* on X.

A set X equipped with an L-pseudo-metric (L-metric) ϱ on X is called an L-pseudo-metric (*L-metric*) *space*.

To each L-pseudo-metric (L-metric) ϱ on a set X is generated canonically a stratified L-topology τ_ϱ on X which has $\{\varepsilon \circ \varrho_x | \varepsilon \in \mathcal{E}, x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}_L^*$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} = \{\bar{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*} | \delta > 0, \alpha \in L\} \cup \{\bar{\alpha} | \alpha \in L\},$$

here $\bar{\alpha}$ has $\mathbf{R}_L^*$ as domain.

An L-topological space (X, τ) is called *pseudo-metrizable* (*metrizable*) if there is an L-pseudo-metric (L-metric) ϱ on X inducing τ, that is, $\tau = \tau_\varrho$.

An L-pseudo-metric ϱ is called *left* (*right*) *invariant* if

$$\varrho(x, y) = \varrho(ax, ay)(\varrho(x, y) = \varrho(xa, ya)) \text{ for all } a, x, y \in X.$$

An L-set $\lambda \in L^X$ is called *countable* (*finite*) if its support is countable (finite), where the support of λ is the set $\{x \in X | 0 < \lambda(x)\}$.

Let us call an L-filter $\mathcal{M}$ on a set X *countable* if the sets $\alpha - \mathrm{pr}\mathcal{M}$ are countable for all $\alpha \in L_0$.

Definition 3.1. An L-topological space (X, τ) is called *first countable* if every point $x \in X$ has a countable L-neighborhood filter $\mathcal{N}(x)$.

Proposition 3.1. *For any L-pseudo-metric ϱ on a set X, if τ_ϱ is the L-topology associated with ϱ, then (X, τ_ϱ) is a first countable space.*

Proof. Since $\{\varepsilon \circ \varrho_x | \varepsilon \in \mathcal{E}, x \in X\}$ is a base for τ_ϱ, then for all $n \in \mathbf{N}$, the set $B_n = \{\varepsilon_n \circ \varrho_x | \varepsilon_n \in \mathcal{E}, x \in X\}$, where $\varepsilon_n = \overline{\frac{1}{n}} \wedge R^\delta|_{\mathbf{R}_L^*}$, is the $\frac{1}{n} - \mathrm{pr}\,\mathcal{N}(x)$, which implies that there exists a countable L-neighborhood filter $\mathcal{N}(x)$ at every point $x \in X$. Hence, (X, τ_ϱ) is a first countable space. □

By an *L-function family* Φ on a set X, we mean the set of all L-real functions f: $X \to I_L$.

We also have the following results.

Lemma 3.1. *Let Φ be an L-function family on X and σ_f: $X \times X \to I_L$ a mapping defined by*

$$\sigma_f(x, y) = (f(x) - f(y))^+, \quad f \in \Phi.$$

Then σ_f is an L-pseudo-metric on X.

Proof. Clearly, $\sigma_f(x, y) = \sigma_f(y, x)$. From (2.1), we get that $\sigma_f(x, x) = (f(x) - f(x))^+ = 0^\sim$ for all $x \in X$, and moreover

$$\begin{aligned}\sigma_f(x, y) &= (f(x) - f(y))^+ \leqslant (f(x) - f(z))^+ + (f(z) - f(y))^+ \\ &= \sigma_f(x, z) + \sigma_f(z, y).\end{aligned}$$

Hence, σ_f is an L-pseudo-metric on X. □

Lemma 3.2. *Let σ_i: $X \times X \to I_L$, $i \in I$ be an arbitrary set of L-pseudo-metrics on the set X. Then*

$$\sigma(x, y) = \sup\{\sigma_i(x, y) | i \in I\}$$

defines an L-pseudo-metric on X as well.

Proof. Only the triangle inequality has to be shown. For all x, y, $z \in X$ and all $i \in I$, we have

$$\sigma_i(x, y) \leqslant \sigma_i(x, z) + \sigma_i(z, y) \leqslant \sigma(x, z) + \sigma(z, y),$$

and then $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y)$. Hence, σ is an L-pseudo-metric on X. □

Here, we have shown this fact.

Lemma 3.3. *Any L-pseudo-metric ϱ on a set X is an L-metric on X* if and only if *(X, τ_ϱ) is a GT_0-space.*

Proof. Let x, $y \in X$ and $y \neq x$. Since (X, τ_ϱ) is a GT_0-space, then there exists $\mu \in L^X$ such that $\mu(x) < \beta \leqslant \mathrm{int}_{\tau_\varrho}\mu(y)$ for some $\beta \in L_0$. From the definition of the base of τ_ϱ, since

$$\mathrm{int}_{\tau_\varrho}\mu(z) = \bar{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*}(\varrho(x, z)) = \alpha \wedge (\bigvee_{\eta \geqslant \delta} \varrho(x, z)(\eta))'$$

for all $z \in X$ and for some $\alpha \in L$, then $\varrho(x, y) = 0^\sim$ implies that $\mathrm{int}_{\tau_\varrho}\mu(y) = \alpha \wedge 1 = \alpha$ for all $y \in X$ and all $\mu \in L^X$. Hence,

$$\alpha = \mathrm{int}_{\tau_\varrho}\mu(x) \leqslant \mu(x) < \beta \leqslant \mathrm{int}_{\tau_\varrho}\mu(y) = \alpha,$$

that is, $\alpha < \beta \leqslant \alpha$ which is a contradiction, and thus $x = y$ and ϱ is an L-metric.

Now let $x \neq y$ and so $\varrho(x, y) \neq 0^\sim$, then there exists $\alpha > 0$ such that $\varrho(x, y)(\alpha) > 0$ and hence taking $v = \bar{1} \wedge R^\delta|_{\mathbf{R}_L^*} \circ \varrho_x \in L^X$, we get that

$$v(y) = 1 \wedge R^\delta(\varrho(x, y)) = 1 \wedge (\bigvee_{\eta \geqslant \delta} \varrho(x, z)(\eta))' < 1$$

whenever δ is chosen to be a very small number tends to zero. But $\mathrm{int}_{\tau_\varrho} v(x) = 1 \wedge (\bigvee_{\eta \geqslant \delta} \varrho(x, x)(\eta))' = 1$. Hence, (X, τ_ϱ) is a GT_0-space. □

4. On L-uniform spaces

An L-filter $\mathcal{U}$ on $X \times X$ is called L-uniform structure on X [5] if the following conditions are fulfilled:

(U1) $(x,x)^{\cdot} \succ \mathcal{U}$ for all $x \in X$;

(U2) $\mathcal{U} = \mathcal{U}^{-1}$;
(U3) $\mathcal{U} \circ \mathcal{U} \succ \mathcal{U}$.

Where $(x,\ x)^{\cdot}(u) = u(x,\ x)$, $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $(\mathcal{U} \circ \mathcal{U})(u) = \bigvee_{v \circ w \leqslant u}(\mathcal{U}(w) \wedge \mathcal{V}(v))$ for all $u \in L^{X \times X}$, and $u^{-1}(x,\ y) = u(y,\ x)$ and $(v \circ w)(x,\ y) = \bigvee_{z \in X}(w(x,\ z) \wedge v(z,\ y))$ for all $x,\ y \in X$.

A set X equipped with an L-uniform structure $\mathcal{U}$ is called an L-uniform space. A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between L-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is said to be *L-uniformly continuous* (or *$(\mathcal{U},\mathcal{V})$-continuous*) provided

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \succ \mathcal{V}$$

holds. To each L-uniform structure $\mathcal{U}$ on X is associated a stratified L-topology $\tau_{\mathcal{U}}$. The related interior operator $\mathrm{int}_{\mathcal{U}}$ is given by:

$$(\mathrm{int}_{\mathcal{U}}\lambda)(x) = \mathcal{U}[\dot{x}](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[\dot{x}](\lambda) = \bigvee_{u[\mu] \leqslant \lambda}(\mathcal{U}(u) \wedge \mu(x))$ and $u[\mu](x) = \bigvee_{y \in X}(\mu(y) \wedge u(y,\ x))$. For all $x \in X$ and all $\lambda \in L^X$ we have

$$\mathcal{U}[\dot{x}] = \mathcal{N}(x) \quad \text{and} \quad \mathcal{U}[\dot{\lambda}] = \mathcal{N}(\lambda),$$

where $\mathcal{N}(x)$ and $\mathcal{N}(\lambda)$ are the L-neighborhood filters of the space $(X, \tau_{\mathcal{U}})$ at x and λ, respectively.

Let $\mathcal{U}$ be an L-uniform structure on a set X. Then $u \in L^{X \times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geqslant \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$. A surrounding $u \in L^{X \times X}$ is called *left* (*right*) *invariant* provided

$$u(ax, ay) = u(x,y)(u(xa, ya) = u(x,y)) \text{ for all } a,x,y \in X.$$

$\mathcal{U}$ is called a *left* (*right*) *invariant* L-uniform structure if $\mathcal{U}$ has a valued L-filter base consists of left (right) invariant surroundings [4].

L-topological groups. In the following we focus our study on a multiplicative group G. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}. Let G be a group and τ an L-topology on G. Then (G, τ) will be called an *L-topological group* [2] if the mappings

$$\pi : (G \times G, \tau \times \tau) \to (G, \tau) \text{ defined by } \pi(a,b) = ab \text{ for all } a,b \in G$$

and

$$i : (G, \tau) \to (G, \tau) \text{ defined by } i(a) = a^{-1} \text{ for all } a \in G$$

are L-continuous. π and i are the binary operation and the unary operation of the inverse on G, respectively.

For all $\lambda \in L^G$, the inverse λ^i of λ with respect to the unary operation i on G is the L-set $\lambda \circ i$ in G defined by [4]

$$\lambda^i(x) = \lambda(x^{-1}) \text{ for all } x \in G.$$

Example 4.1. *For a group G, the induced L-topological space $(G,\ \omega_L(T))$ of the usual topological group $(G,\ T)$ is an L-topological group.*

Proposition 4.1. [4] *Let $(G,\ \tau)$ be an L-topological group. Then there exist on G a unique left invariant L-uniform structure $\mathcal{U}^l$ and a unique right invariant L-uniform structure $\mathcal{U}^r$ compatible with τ, constructed using the family $(\alpha - \mathrm{pr}\ \mathcal{N}(e))_{\alpha \in L_0}$ of all filters $\alpha - \mathrm{pr}\ \mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L-neighborhood filter at the identity element e of $(G,\ \tau)$, as follows:*

$$\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}^l_\alpha, v \leqslant u} \alpha \quad \text{and} \quad \mathcal{U}^r(u) = \bigvee_{v \in \mathcal{U}^r_\alpha, v \leqslant u} \alpha, \tag{4.1}$$

where

$$\mathcal{U}^l_\alpha = \alpha - \mathrm{pr}\ \mathcal{U}^l \quad \text{and} \quad \mathcal{U}^r_\alpha = \alpha - \mathrm{pr}\ \mathcal{U}^r \tag{4.2}$$

are defined by

$$\mathcal{U}^l_\alpha = \{u \in L^{G \times G} | u(x,y) = (\lambda \wedge \lambda^i)(x^{-1}y) \text{ for some } \lambda \in \alpha - \mathrm{pr}\ \mathcal{N}(e)\} \tag{4.3}$$

and

$$\mathcal{U}^r_\alpha = \{u \in L^{G \times G} | u(x,y) = (\lambda \wedge \lambda^i)(xy^{-1}) \text{ for some } \lambda \in \alpha - \mathrm{pr}\ \mathcal{N}(e)\} \tag{4.4}$$

We should notice that we shall fix the notations $\mathcal{U}^l$, $\mathcal{U}^r$, $\mathcal{U}^l_\alpha$ and $\mathcal{U}^r_\alpha$ along the paper to be these defined above.

Remark 4.1. For the L-topological group (G, τ), the elements u of $\mathcal{U}^l_\alpha(\mathcal{U}^r_\alpha)$ are left (right) invariant surroundings. Moreover, $(\mathcal{U}^l_\alpha)_{\alpha \in L_0}((\mathcal{U}^r_\alpha)_{\alpha \in L_0})$ is a valued L-filter base for the left (right) invariant L-uniform structure $\mathcal{U}^l(\mathcal{U}^r)$ defined by (4.1), (4.2), (4.3), (4.4), respectively.

L-topogenous orders. A binary relation on L^X is said to be an *L-topogenous order* on X [13] if the following conditions are fulfilled:

(1) $\bar{0} \ll \bar{0}$ and $\bar{1} \ll \bar{1}$;
(2) λ μ implies $\lambda \leqslant \mu$;
(3) $\lambda_1 \leqslant \lambda$ $\mu \leqslant \mu_1$ implies λ_1 μ_1;
(4) From λ_1 μ_1 and λ_2 μ_2 it follows $\lambda_1 \vee \lambda_2$ $\mu_1 \vee$ μ_2 and $\lambda_1 \wedge \lambda_2$ $\mu_1 \wedge \mu_2$.

An L-topogenous order is said to be *regular* or is said to be an L-topogenous structure if for all $\lambda,\ \mu \in L^X$ with λ μ there is a $v \in L^X$ such that λ v and v μ hold, and is called *complementarily symmetric* if λ μ implies μ' λ' for all $\lambda,\ \mu \in L^X$ and moreover is called *perfect* if for each family $(\lambda_i)_{i \in I}$ of L-subsets of X with λ_i μ for all $i \in I$ it follows $\bigvee_{i \in I} \lambda_i \ll \mu$.

Let $(_n)$ be a sequence of L-topogenous structures on X and $(\prec_n)$ a sequence of L-topogenous structures on I_L. Then an L-real function $f\colon X \to I_L$ is said to be *associated with* the sequence $(_n)$ if for all $\lambda, \mu \in L^{I_L}$, $\lambda \prec_n \mu$ implies $(\lambda \circ f)$ $_{n+1}(\mu \circ f)$ for every positive integer n [11].

Now, suppose that $(G,\ \tau)$ has a countable L-neighborhood filter $\mathcal{N}(e)$ at the identity e. Since any L-topological group, from Proposition 4.1, is uniformizable, then the left and the right invariant L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$, constructed also in Proposition 4.1, has, from Remark 4.1, a countable L-filter base $\mathcal{U}^l_{\frac{1}{n}}$ and $\mathcal{U}^r_{\frac{1}{n}}$, respectively, $n \in \mathbf{N}$.

Lemma 4.1. [18] *For all $\lambda,\ \mu \in L^X$, we have*

$\lambda \leqslant \mu$ if and only if $\dot{\lambda} \succ \dot{\mu}$.

Here, we prove this interesting result.

Lemma 4.2. *Let $\mathcal{U}$ be an L-uniform structure on a set X, and define a binary relation on L^X as follows:*

$$\lambda \ll_{\mathcal{U}} \mu \Longleftrightarrow \mathcal{U}[\dot{\lambda}] \succ \dot{\mu}$$

for all $\lambda, \mu \in L^X$. Then $\ll_{\mathcal{U}}$ is a complementarily symmetric perfect L-topogenous order on X.

Proof. From the properties of $\mathcal{U}$ as an L-filter, (2.1) and Lemma 4.1 we get easily that $\ll_{\mathcal{U}}$ fulfills all the required conditions. □

Proposition 4.2. [13] *There is a one-to-one correspondence between the perfect L-topogenous structures on a set X and the L-topologies τ on X. This correspondence is given by*

$$\lambda \ll \mu \Longleftrightarrow \lambda \leqslant \nu \leqslant \mu \text{ for some } \nu \in \tau$$

for all $\lambda, \mu \in L^X$ and

$$\tau = \{\lambda \in L^X | \lambda \ll \lambda\}.$$

Now we have the following result.

Proposition 4.3. *Suppose that $\mathcal{U}$ and $\left(\mathcal{U}_{\frac{1}{n}}\right)_{n\in N}$ are an L-uniform structure on X and its countable L-filter base, respectively, and also consider $\mathcal{V}$ an L-uniform structure on I_L. Let $(_n)_{n\in\mathbf{N}}$ denote a sequence of complementarily symmetric perfect L-topogenous structures on X for which $\lambda \ll_n \mu \Longleftrightarrow \mathcal{U}[\dot{\lambda}] \succ \dot{\mu}$ for all $\lambda, \mu \in L^X$, and let Φ be the family of all L-uniformly continuous functions $h : (X,\mathcal{U}) \to (I_L, \mathcal{V})$ associated with $(_n)_{n\in\mathbf{N}}$. Then the mapping $\sigma_{\mathcal{U}} : X \times X \to I_L$ defined by*

$$\sigma_{\mathcal{U}}(x, y) = \sup\ \{\sigma_f(x, y) | f \in \Phi\},$$

where $\sigma_f(x, y) = (f(x) - f(y))^+$ for all $x, y \in X$, is an L-pseudo-metric on X and $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$.

Proof. The proof of that $\sigma_{\mathcal{U}}$ is an L-pseudo-metric on X comes from Lemmas 3.1, 3.2, and 4.2.

Since for any $\lambda \in L^X$, and from Proposition 4.2

$$\lambda \ll_n \lambda \Longleftrightarrow \mathcal{U}[\dot{\lambda}] \succ \dot{\lambda}$$

means that $\lambda \in \tau_{\mathcal{U}}$ if and only if $\lambda \in \tau_{\sigma_{\mathcal{U}}}$, where $\sigma_{\mathcal{U}}$ is generated by all the L-pseudo-metrics σ_h for every h associated with $_n$. Hence, $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$. □

5. The metrizability of L-topological groups

This section is devoted to show that any (separated) L-topological group is pseudo-metrizable (metrizable).

An L-topological group (G, τ) is called *separated* if for the identity element e, we have $\bigwedge_{\lambda\in\alpha-\mathrm{pr}\mathcal{N}(e)} \lambda(e) \geqslant \alpha$, and $\bigwedge_{\lambda\in\alpha-\mathrm{pr}\mathcal{N}(e)} \lambda(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$ [4].

Proposition 5.1. [4] *Any* (separated) *L-topological group is a ($GT_{3\frac{1}{2}}$-space) completely regular space.*

Now, we are going to show the main result in this paper.

Proposition 5.2. *Any (separated) L-topological group (G, τ)* is pseudo-metrizable (metrizable).

Proof. From Proposition 4.1, we have unique left and unique right L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G defined by (4.1) such that $\tau = \tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$. Proposition 4.3 implies that $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$ and $\tau = \tau_{\mathcal{U}^r} = \tau_{\sigma_{\mathcal{U}^r}}$, and therefore (G, τ) is pseudo-metrizable.

Also, if (G, τ) is separated, then from Proposition 5.1, we get that (G, τ) is a GT_0-space, and hence, from Lemma 3.3, we have that (G, τ) is metrizable. □

We also have the following important result.

Proposition 5.3. *Let (G, τ) be a (separated) L-topological group. Then the following statements are equivalent.*

(1) *τ is pseudo-metrizable (metrizable);*
(2) *e has a countable L-neighborhood filter $\mathcal{N}(e)$;*
(3) *τ can be induced by a left invariant L-pseudo-metric (L-metric);*
(4) *τ can be induced by a right invariant L-pseudo-metric (L-metric).*

Proof. (1) ⇒ (2): Follows from Proposition 3.1

(2) ⇒ (3): Let e has a countable L-neighborhood filter $\mathcal{N}(e)$, and let $\mathcal{U}^l_{\frac{1}{n}}$ be a countable L-filter base of the left invariant L-uniform structure $\mathcal{U}^l$, defined by (4.1), which is compatible with τ. Then, from Lemma 4.2, $\lambda \ll_{\mathcal{U}^l} \mu \Longleftrightarrow \mathcal{U}^l[\dot{\lambda}] \succ \dot{\mu}$ for all $\lambda, \mu \in L^G$ defines a sequence of complementarily symmetric perfect L-topogenous structures on G. Taking $\mathcal{V}$ as an L-uniform structure on I_L and Φ as the family of all L-uniformly continuous functions $h : (G,\mathcal{U}^l) \to (I_L, \mathcal{V})$ associated with $\ll_{\mathcal{U}^l}$, we get, from Proposition 4.3, that the L-mapping σ: $G \times G \to I_L$ defined by $\sigma(x, y) = \sup\{(f(x) - f(y))^+ | f \in \Phi\}$ is an L-pseudo-metric on G and $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$.

Now, we define h_a: $G \to I_L$ by $h_a(x) = h(a\ x)$ for all $a, x \in G$. From $h \in \Phi$ is L-uniformly continuous, that is, $\mathcal{F}_L(h \times h)(\mathcal{U}^l) \succ \mathcal{V}$ and that the elements of $\mathcal{U}^l_{\frac{1}{n}}$ are left invariant from Remark 4.1, and from (4.1), we have

$$\begin{aligned}
\mathcal{F}_L(h_a \times h_a)\mathcal{U}^l(v) &= \mathcal{U}^l(v \circ (h_a \times h_a)) \\
&= \bigvee_{u\in\mathcal{U}^l_{\frac{1}{n}}, u\leqslant v\circ(h_a\times h_a)} \frac{1}{n} \\
&= \bigvee_{u\in\mathcal{U}^l_{\frac{1}{n}}, u\leqslant v\circ(h\times h)} \frac{1}{n} \\
&= \mathcal{F}_L(h \times h)\mathcal{U}^l(v) \\
&\geqslant \mathcal{V}(v).
\end{aligned}$$

Hence, h_a is L-uniformly continuous associated with $\ll_{\mathcal{U}^l}$, that is, $h_a \in \Phi$. Thus

$$\begin{aligned}
\sigma(ax, ay) &= \sup\{(h(ax) - h(ay))^+ | h \in \Phi\} \\
&= \sup\{(h_a(x) - h_a(y))^+ | h \in \Phi\} \\
&\leqslant \sup\ \{(k(x) - k(y))^+ | k \in \Phi\} \\
&= \sigma(x, y).
\end{aligned}$$

Applying the same for a^{-1} instead of a, we get that $\sigma(x, y) = \sigma(a^{-1}a \ x, \ a^{-1}a \ y) \leqslant \sigma(a \ x, \ a \ y)$. That is, $\sigma(a \ x, \ a \ y) = \sigma(x, y)$ for all $a, x, y \in G$ and then σ is a left invariant L-pseudo-metric on G inducing τ.

(2) $\Rightarrow$ (4): By a similar proof as in the case (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1): Obvious.

The proposition is still true if we consider the parentheses. □

Example 5.1. From Proposition 5.2, we can deduce that any L-topological group (G, τ) on which there can be constructed an L-uniform structure $\mathcal{U}$ compatible with τ is pseudo-metrizable in general and is metrizable whenever (G, τ) is separated.

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